

SOME FIXED POINT THEOREMS ON S -METRIC SPACES

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Abstract. In this paper, we present some contractive mappings and prove new generalized fixed point theorems on S -metric spaces. Also we define the notion of a cluster point and investigate fixed points of self-mappings using cluster points on S -metric spaces. We obtain new generalizations of the classical Nemytskii-Edelstein and Ćirić's fixed point theorems for continuous self-mappings of compact S -metric spaces.

1. Introduction

Metric spaces are very important in various areas of mathematics such as analysis, topology, applied mathematics etc. So various generalizations of metric spaces have been studied and several fixed point results were obtained (for example, see [4, 7, 8, 11–15]). Recently, Sedghi, Shobe and Aliouche have defined the concept of an S -metric space as follows:

DEFINITION 1.1. [12] Let X be a nonempty set and $S : X^3 \rightarrow [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$:

- (1) $S(x, y, z) = 0$ if and only if $x = y = z$,
- (2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then S is called an S -metric on X and the pair (X, S) is called an S -metric space.

Let (X, d) be a complete metric space and T be a self-mapping on X . In [10], the following condition was introduced for a self-mapping T : for each $x, y \in X$, $x \neq y$:

$$(R25) \quad d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

(such mappings were later called Rhoades' mappings). However, no fixed point theorem was given in [10] for mappings satisfying (R25). Chang presented the notion of a C -mapping and obtained some fixed point theorems using such mappings

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under (R25) in [1]. Also, Liu, Xu and Cho defined the concept of an L -mapping and studied some fixed point theorems using such mappings under (R25) in [5]. On the other hand, Park introduced a new contractive mapping using the diameter of $U_x \cup U_y$ where $U_x = \{T^n x : n \in \mathbb{N}\}$ in [9]. He presented also the relationships between these contractive mappings and the condition (R25) and then obtained some fixed point theorems.

Motivated by the above studies, we extend the notion of Rhoades' mapping to S -metric spaces and define a new type of contractive mappings. In Section 2, we introduce new contractive conditions (S25) and (S25a), defining the notions of a C_S -mapping and an L_S -mapping on S -metric spaces. Also, we investigate some relations among them and give some counterexamples. In Section 3, we prove some fixed point theorems using the notions of a C_S -mapping, an L_S -mapping, a periodic point and compactness on S -metric spaces. In Section 4, we present the notion of a cluster point on an S -metric space and study some properties of cluster points. We give some fixed point theorems by means of cluster points on S -metric spaces. In Section 5, we obtain new generalizations of the classical Nemytskii-Edelstein and Ćirić's fixed point theorems for continuous self-mappings on a compact S -metric space.

2. Contractive mappings on S -metric spaces

In this section, we define some new contractive mappings and the notions of a C_S -mapping and an L_S -mapping on an S -metric space. Also we investigate their relationships with each other and give counterexamples.

Now we recall some definitions, lemmas, a remark and a corollary which are needed in the sequel. The following can be found in the papers referred to.

DEFINITION 2.1. [12] Let (X, S) be an S -metric space and $A \subset X$.

- (1) A subset A of X is called S -bounded if there exists $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$.
- (2) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. That is, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$ for each $\varepsilon > 0$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $\lim_{n \rightarrow \infty} S(x_n, x_n, x) = 0$.
- (3) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$. That is, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $S(x_n, x_n, x_m) < \varepsilon$ for each $\varepsilon > 0$.
- (4) The S -metric space (X, S) is called complete if every Cauchy sequence is convergent.

LEMMA 2.1. [12] Let (X, S) be an S -metric space. Then

$$S(x, x, y) = S(y, y, x), \quad (2.1)$$

for all $x, y \in X$.

LEMMA 2.2. [12] *Let (X, S) be an S -metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \rightarrow x$ and $y_n \rightarrow y$, then $S(x_n, x_n, y_n) \rightarrow S(x, x, y)$.*

REMARK 2.1. [13] Every S -metric space is topologically equivalent to a B -metric space.

COROLLARY 2.1. [13] *Let $T : X \rightarrow Y$ be a map from an S -metric space X to an S -metric space Y . Then T is continuous at $x \in X$ if and only if $Tx_n \rightarrow Tx$ whenever $x_n \rightarrow x$.*

Now we consider the Rhoades' condition (R25) for S -metric spaces and define a C_S -mapping (or an L_S -mapping).

DEFINITION 2.2. Let (X, S) be an S -metric space and T be a self-mapping of X . We define

$$(S25) \quad S(Tx, Tx, Ty) < \max\{S(x, x, y), S(Tx, Tx, x), S(Ty, Ty, y), \\ S(Ty, Ty, x), S(Tx, Tx, y)\},$$

for each $x, y \in X$, $x \neq y$.

DEFINITION 2.3. Let (X, S) be an S -metric space and T be a self-mapping on X . T is called a C_S -mapping on X if for each $x \in X$ and each positive integer $n \geq 2$ satisfying

$$T^i x \neq T^j x, 0 \leq i < j \leq n - 1, \quad (2.2)$$

we have

$$S(T^n x, T^n x, T^i x) < \max_{1 \leq j \leq n} \{S(T^j x, T^j x, x)\}, \quad i = 1, 2, \dots, n - 1. \quad (2.3)$$

DEFINITION 2.4. Let (X, S) be an S -metric space and T be a self-mapping on X . T is called an L_S -mapping on X if for each $x \in X$ and each positive integer $n \geq 2$ with the condition (2.2) we have

$$S(T^n x, T^n x, T^i x) < \max_{0 \leq p < q \leq n} \{S(T^p x, T^p x, T^q x)\}, \quad i = 1, 2, \dots, n - 1. \quad (2.4)$$

PROPOSITION 2.1. *Let (X, S) be an S -metric space and T be a self-mapping on X . If T satisfies the condition (S25), then T is a C_S -mapping.*

Proof. Let $x \in X$ and the condition (S25) be satisfied by T . We use the mathematical induction. Assume that the condition (2.2) holds for each $n \geq 2$. For $n = 2$, by (S25) we have

$$S(T^2 x, T^2 x, Tx) < \max\{S(Tx, Tx, x), S(T^2 x, T^2 x, Tx), S(Tx, Tx, x), \\ S(Tx, Tx, Tx), S(T^2 x, T^2 x, x)\} \quad (2.5)$$

and so

$$S(T^2 x, T^2 x, Tx) < \max\{S(Tx, Tx, x), S(T^2 x, T^2 x, x)\}.$$

Hence the condition (2.3) is satisfied.

Suppose that the condition (2.3) is true for $n = k - 1$, $k \geq 3$. Let

$$\alpha = \max_{1 \leq j \leq k-1} \{S(T^j x, T^j x, x)\}.$$

We show that the condition (2.3) is satisfied for $n = k$, $k \geq 2$. By the condition (S25) and the induction hypothesis we find

$$\begin{aligned} S(T^k x, T^k x, T^{k-1} x) &< \max\{S(T^{k-1} x, T^{k-1} x, T^{k-2} x), S(T^k x, T^k x, T^{k-1} x), \\ &S(T^{k-1} x, T^{k-1} x, T^{k-2} x), S(T^{k-1} x, T^{k-1} x, T^{k-1} x), \\ &S(T^k x, T^k x, T^{k-2} x)\} \end{aligned}$$

and so

$$S(T^k x, T^k x, T^{k-1} x) < \max\{\alpha, S(T^k x, T^k x, T^{k-2} x)\}.$$

Also it can be shown that

$$S(T^k x, T^k x, T^{k-i} x) < \max\{\alpha, S(T^k x, T^k x, T^{k-i-1} x)\}, \quad i = 1, 2, \dots, k-1.$$

For $i = k-1$ we obtain

$$S(T^k x, T^k x, T x) < \max\{\alpha, S(T^k x, T^k x, x)\} = \max_{1 \leq j \leq k} \{S(T^j x, T^j x, x)\}$$

and

$$S(T^k x, T^k x, T^i x) < \max_{1 \leq j \leq k} \{S(T^j x, T^j x, x)\}, \quad i = 1, 2, \dots, k-1.$$

Hence the condition (2.3) is satisfied. The proof is completed. ■

The converse of Proposition 2.1 is not always true as we see in the following example.

EXAMPLE 2.1. Let \mathbb{R} be the real line. Let us consider the usual S -metric on \mathbb{R} defined in [13] as follows

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in \mathbb{R}$. Let

$$Tx = \begin{cases} x, & \text{if } x \in [0, 1] \\ x - 4, & \text{if } x = 6, 10 \\ 1, & \text{if } x = 2 \end{cases}$$

Then T is a self-mapping on the S -metric space $[0, 1] \cup \{2, 6, 10\}$.

For $x = \frac{1}{2}$, $y = \frac{1}{3} \in [0, 1]$ we have

$$\begin{aligned} S(Tx, Tx, Ty) &= S\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}\right) = \frac{1}{3}, & S(x, x, y) &= S\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}\right) = \frac{1}{3}, \\ S(Tx, Tx, x) &= S(x, x, x) = 0, & S(Ty, Ty, y) &= S(y, y, y) = 0, \\ S(Ty, Ty, x) &= S\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}\right) = \frac{1}{3}, & S(Tx, Tx, y) &= S\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}\right) = \frac{1}{3} \end{aligned}$$

and so

$$S(Tx, Tx, Ty) = \frac{1}{3} < \max\{\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}\} = \frac{1}{3}.$$

Hence T does not satisfy the condition (S25).

We now show that T is a C_S -mapping. We have the following cases for $x \in \{2, 6, 10\}$.

Case 1. For $x = 2$ and $n = 2$ we have

$$S(T^2 2, T^2 2, T2) = 0 < \max\{S(T^2 2, T^2 2, 2), S(T2, T2, 2)\} = 2.$$

For $n > 2$ using similar arguments we can see that (2.3) holds.

Case 2. For $x = 6$ and $n \in \{2, 3\}$ we have

$$S(T^2 6, T^2 6, T6) = 2 < \max\{S(T^2 6, T^2 6, 6), S(T6, T6, 6)\} = 10$$

and

$$\begin{aligned} \max\{S(T^3 6, T^3 6, T6), S(T^3 6, T^3 6, T^2 6)\} &= 2 \\ &< \max\{S(T^3 6, T^3 6, 6), S(T^2 6, T^2 6, 6), S(T6, T6, 6)\} = 10. \end{aligned}$$

For $n > 3$ using similar arguments we can see that (2.3) holds.

Case 3. For $x = 10$ and $n \in \{2, 3, 4\}$ we have

$$S(T^2 10, T^2 10, T10) = 8 < \max\{S(T^2 10, T^2 10, 10), S(T10, T10, 10)\} = 16,$$

$$\begin{aligned} \max\{S(T^3 10, T^3 10, T10), S(T^3 10, T^3 10, T^2 10)\} &= 10 \\ &< \max\{S(T^3 10, T^3 10, 10), S(T^2 10, T^2 10, 10), S(T10, T10, 10)\} = 18 \end{aligned}$$

and

$$\begin{aligned} \max\{S(T^4 10, T^4 10, T10), S(T^4 10, T^4 10, T^2 10), S(T^4 10, T^4 10, T^3 10)\} &= 10 \\ &< \max\{S(T^4 10, T^4 10, 10), S(T^3 10, T^3 10, 10), S(T^2 10, T^2 10, 10), S(T10, T10, 10)\} \\ &= 18. \end{aligned}$$

For $n > 4$ using similar arguments we can see that (2.3) holds. Hence T is a C_S -mapping.

PROPOSITION 2.2. *Let (X, S) be an S -metric space. Then the notions of a C_S -mapping and an L_S -mapping are equivalent.*

Proof. Let T be an L_S -mapping and $x \in X$. Suppose that the condition (2.2) is satisfied for each positive integer $n \geq 2$. Then we have

$$\min\{S(T^i x, T^i x, T^j x) : 0 \leq i < j \leq k - 1\} > 0,$$

where $2 \leq k \leq n$. Let

$$\alpha_n = \max_{1 \leq i \leq n-1} \{S(T^n x, T^n x, T^i x)\} \quad \text{and} \quad \beta_n = \max_{1 \leq i \leq n} \{S(T^i x, T^i x, x)\}.$$

By the conditions (2.1), (2.4) and (2.5) we obtain

$$S(T^n x, T^n x, T^i x) < \max_{0 \leq p < q \leq n} \{S(T^p x, T^p x, T^q x)\},$$

where $i = 1, 2, \dots, n-1$ and

$$\begin{aligned} \alpha_n &= \max\{S(T^n x, T^n x, T^i x) : 1 \leq i \leq n-1\} \\ &< \max\{\alpha_n, \beta_n, \max\{S(T^p x, T^p x, T^q x) : 1 \leq p < q \leq n-1\}\} \\ &= \max\{\beta_n, \max\{S(T^p x, T^p x, T^q x) : 1 \leq p < q \leq n-1\}\} \\ &= \max\{\alpha_{n-1}, \beta_n, \max\{S(T^p x, T^p x, T^q x) : 1 \leq p < q \leq n-2\}\} \\ &\leq \max\{\beta_n, \beta_{n-1}, \max\{S(T^p x, T^p x, T^q x) : 1 \leq p < q \leq n-2\}\} \\ &= \max\{\beta_n, \max\{S(T^p x, T^p x, T^q x) : 1 \leq p < q \leq n-2\}\} \\ &\leq \dots \\ &\leq \max\{\beta_n, \max\{S(T^p x, T^p x, T^q x) : 1 \leq p < q \leq 2\}\} \\ &= \max\{\beta_n, S(Tx, Tx, T^2 x)\} = \max\{\beta_n, S(T^2 x, T^2 x, Tx)\} \\ &\leq \max\{\beta_n, \max\{S(Tx, Tx, x), S(T^2 x, T^2 x, x)\}\} = \beta_n. \end{aligned}$$

Hence the condition (2.3) is satisfied. Consequently T is a C_S -mapping.

Conversely, let T be a C_S -mapping and $x \in X$. Suppose that the condition (2.2) is satisfied for each positive integer $n \geq 2$. We now show that T is an L_S -mapping. From the condition (2.3) we have

$$S(T^n x, T^n x, T^i x) < \max_{1 \leq j \leq n} \{S(T^j x, T^j x, x)\}, \quad i = 1, 2, \dots, n-1.$$

If $1 \leq j \leq n$, then $0 \leq j-1 \leq n-1$. Let q be chosen such that $0 \leq j-1 < q \leq n$. For $j-1 = 0$ we have $1 \leq q \leq n$ and

$$S(T^n x, T^n x, T^i x) < \max_{1 \leq q \leq n} \{S(T^q x, T^q x, x)\}.$$

If we put $j-1 = p$ then we have

$$S(T^n x, T^n x, T^i x) < \max_{0 \leq p < q \leq n} \{S(T^q x, T^q x, T^p x)\} = \max_{0 \leq p < q \leq n} \{S(T^p x, T^p x, T^q x)\}.$$

Consequently T is an L_S -mapping. The proof is completed. ■

Now we give the definition of the notion of diameter on an S -metric space.

DEFINITION 2.5. Let (X, S) be an S -metric space and A be a nonempty subset of X . We define

$$\text{diam}\{A\} = \sup\{S(x, x, y) : x, y \in A\}.$$

Then $\text{diam}\{A\}$ is called the diameter of A . If A is an S -bounded set, then we will write $\text{diam}\{A\} < \infty$.

DEFINITION 2.6. Let (X, S) be an S -metric space, T be a self-mapping on X , $U_x = \{T^n x : n \in \mathbb{N}\}$, $\text{diam}\{U_x\} < \infty$ and $\text{diam}\{U_y\} < \infty$. We define

$$(S25a) \quad S(Tx, Tx, Ty) < \text{diam}\{U_x \cup U_y\},$$

for each $x, y \in X$ with $x \neq y$.

PROPOSITION 2.3. *Let (X, S) be an S -metric space and T be a self-mapping on X . If T satisfies the condition (S25), then T satisfies the condition (S25a).*

Proof. Assume that T satisfies the condition (S25). Then we have

$$\begin{aligned} S(Tx, Tx, Ty) &< \max\{S(x, x, y), S(Tx, Tx, x), S(Ty, Ty, y), \\ &S(Ty, Ty, x), S(Tx, Tx, y)\} \\ &< \text{diam}\{U_x \cup U_y\}. \end{aligned}$$

Consequently T satisfies the condition (S25a). ■

The converse of Proposition 2.3 is not always true as we can see in the following example.

EXAMPLE 2.2. Let the function $S : X^3 \rightarrow [0, \infty)$ be the usual S -metric on \mathbb{R} given in Example 2.1. We define

$$Tx = x, x \in (0, 1) \quad \text{and} \quad S_1(x, y, z) = \frac{S(x, y, z)}{2}.$$

Then clearly $S_1(x, y, z)$ is an S -metric on \mathbb{R} .

For $x = \frac{1}{2}, y = \frac{1}{4} \in (0, 1)$ we have

$$\begin{aligned} S_1(Tx, Tx, Ty) &= S_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right) = \frac{1}{4}, & S_1(x, x, y) &= S_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right) = \frac{1}{4}, \\ S_1(Tx, Tx, x) &= S_1(x, x, x) = 0, & S_1(Ty, Ty, y) &= S_1(y, y, y) = 0, \\ S_1(Ty, Ty, x) &= S_1\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4}, & S_1(Tx, Tx, y) &= S_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right) = \frac{1}{4} \end{aligned}$$

and so we obtain

$$S_1(Tx, Tx, Ty) = \frac{1}{4} < \max\left\{\frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}\right\} = \frac{1}{4}.$$

Therefore T does not satisfy the condition (S25). It can be easily seen that T satisfies the condition (S25a) since $\sup\{(0, 1)\} = 1$.

3. Some fixed point theorems on S -metric spaces

In this section, we present some fixed point theorems using the notions of a C_S -mapping, an L_S -mapping, compactness and diameter on S -metric spaces.

THEOREM 3.1. *Let T be a C_S -mapping from an S -metric space (X, S) into itself. Then T has a fixed point in X if and only if there exist integers p and q , $p > q \geq 0$ and $x \in X$ satisfying*

$$T^p x = T^q x. \tag{3.1}$$

If the condition (3.1) is satisfied, then $T^q x$ is a fixed point of T .

Proof. Let $x_0 \in X$ be a fixed point of T , that is, $Tx_0 = x_0$. For $p = 1, q = 0$ the condition (3.1) is satisfied.

Conversely, suppose that there exist integers p and $q, p > q \geq 0$ and $x \in X$ satisfying (3.1). Let p be the minimal integer such that $T^k x = T^p x, k > q$. If we put $T^q x = y, n = p - q$ we have

$$T^n y = T^n T^q x = T^{p-q+q} x = T^p x = T^q x = y$$

and n is the minimal integer such that $T^n y = y, n \geq 1$.

We now show that y is a fixed point of T . Assume that y is not a fixed point of T . Then $n \geq 2$, and

$$T^i y \neq T^j y$$

for $0 \leq i < j \leq n - 1$. Since T is a C_S -mapping we have

$$\begin{aligned} S(T^i y, T^i y, y) &= S(T^i y, T^i y, T^n y) = S(T^n y, T^n y, T^i y) < \max_{1 \leq j \leq n} \{S(T^j y, T^j y, y)\} \\ &= \max_{1 \leq j \leq n-1} \{S(T^j y, T^j y, y)\}, \quad i = 1, 2, \dots, n-1. \end{aligned}$$

Then we obtain

$$\max_{1 \leq i \leq n-1} \{S(T^i y, T^i y, y)\} < \max_{1 \leq j \leq n-1} \{S(T^j y, T^j y, y)\}.$$

This is a contradiction. Consequently $T^q x = y$ is a fixed point of T . ■

COROLLARY 3.1. *Let (X, S) be an S -metric space and T be a self-mapping of X satisfying the condition (S25). Then T has a fixed point in X if and only if there exist integers p and $q, p > q \geq 0$ and $x \in X$ satisfying (3.1). If the condition (3.1) is satisfied, then $T^q x$ is a fixed point of T .*

THEOREM 3.2. *Let T be an L_S -mapping from an S -metric space (X, S) into itself. Then T has a fixed point in X if and only if there exist integers p and $q, p > q \geq 0$ and $x \in X$ satisfying (3.1). If the condition (3.1) is satisfied, then $T^q x$ is a fixed point of T .*

Proof. It is obvious from Proposition 2.2 and Theorem 3.1. ■

Now we obtain another fixed point theorem using the notion of periodic index.

DEFINITION 3.1. [2] Let (X, S) be an S -metric space, T be a self-mapping on X and $x \in X$. A point x is called a periodic point of T , if there exists a positive integer n such that

$$T^n x = x. \tag{3.2}$$

The least positive integer satisfying the condition (3.2) is called the periodic index of x .

THEOREM 3.3. *Let T be an L_S -mapping from an S -metric space (X, S) into itself. Then T has a fixed point in X if and only if T has a periodic point in X .*

Proof. Let $x_0 \in X$ be a fixed point of T , that is, $Tx_0 = x_0$. For $n = 1$, the condition (3.2) is satisfied. Therefore T has a periodic point x_0 in X .

Conversely, suppose that $x_0 \in X$ is a periodic point of T , that is, there exists a positive integer n such that

$$T^n x_0 = x_0.$$

We now show that x_0 is a fixed point of T . To the contrary, assume that x_0 is not a fixed point of T . Then $n \geq 2$, and

$$T^i x_0 \neq T^j x_0$$

for $0 \leq i < j \leq n - 1$. Since T is an L_S -mapping,

$$S(T^n x_0, T^n x_0, T^i x_0) < \max_{0 \leq p < q \leq n} \{S(T^p x_0, T^p x_0, T^q x_0)\}, \quad i = 1, 2, \dots, n - 1.$$

For $q = n$ we have

$$\max_{1 \leq i \leq n-1} \{S(T^i x_0, T^i x_0, T^n x_0)\} < \max_{0 \leq p \leq n-1} \{S(T^p x_0, T^p x_0, T^n x_0)\},$$

which is a contradiction. Consequently x_0 is a fixed point of T . ■

COROLLARY 3.2. *Let (X, S) be an S -metric space, T be a self-mapping on X and T satisfies the condition (S25). Then the following are equivalent:*

- (1) T has a fixed point in X ,
- (2) T has a periodic point in X ,
- (3) There exist integers p and q , $p > q \geq 0$ and $x \in X$ satisfying $T^p x = T^q x$.

If the condition (3) is satisfied, then $T^q x$ is a fixed point of T .

The S -metric space (X, S) is said to be compact if every sequence in X has a convergent subsequence. Now we give a fixed point theorem for compact S -metric spaces.

THEOREM 3.4. *Let T be a continuous self-mapping from a compact S -metric space (X, S) into itself and T satisfies the condition (S25a). Then T has a unique fixed point.*

Proof. Since T is a continuous self-map and X is compact, there exist a compact subset Y of X such that $TX \subset Y$. Then $TY \subset Y$ and $A = \bigcap_{n=1}^{\infty} T^n Y$ is a nonempty compact subset of X which is mapped by T onto itself. We now show that A is a singleton consisting of the unique fixed point x_0 of T . Assume that A is not a singleton. Then we have $\text{diam}\{A\} > 0$. Since A is a compact subset, there exist $x, y \in A$ with $S(x, x, y) = \text{diam}\{A\}$. Also there exist $x', y' \in A$ with $Tx' = x$, $Ty' = y$ since T maps A onto itself. Since T satisfies the condition (S25a) we have

$$\text{diam}\{A\} = S(x, x, y) = S(Tx', Tx', Ty') < \text{diam}\{A\},$$

which is a contradiction. Consequently T has a unique fixed point. ■

COROLLARY 3.3. *Let T be a continuous self-mapping from a compact S -metric space (X, S) into itself and T satisfies the condition (S25). Then T has a unique fixed point.*

4. Fixed point theorems via cluster points

In this section, we obtain new fixed point theorems by means of cluster points on an S -metric space.

DEFINITION 4.1. [12] Let (X, S) be an S -metric space. For $r > 0$ and $x \in X$, the open ball $B_S(x, r)$ is defined as follows:

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\}.$$

DEFINITION 4.2. Let (X, S) be an S -metric space and $A \subset X$ be any subset. A point $x \in X$ is a cluster point of A if

$$(B_S(x, r) - \{x\}) \cap A \neq \emptyset,$$

for every $r > 0$.

THEOREM 4.1. *Let (X, S) be an S -metric space and $A \subset X$. Then x is a cluster point of A if and only if there exist $x_i \in S$ ($i = 1, 2, \dots, n, \dots$) such that $x_i \neq x_j$ for each $i \neq j$ and $\lim_{n \rightarrow \infty} S(x_n, x_n, x) = 0$.*

Proof. Assume that there exist $x_i \in S$ ($i = 1, 2, 3, \dots, n, \dots$) such that $x_i \neq x_j$ for each $i \neq j$ and $\lim_{n \rightarrow \infty} S(x_n, x_n, x) = 0$. Then the sequence $\{x_n\}$ converges to x in $A - \{x\}$. Hence for any $r > 0$, there is $n_0 \in \mathbb{N}$ such that $x_n \in B_S(x, r)$ for $n \geq n_0$. So we obtain $(B_S(x, r) - \{x\}) \cap A \neq \emptyset$. Consequently x is a cluster point of A .

Conversely, let x be a cluster point of A . We choose $x_1 \in A$ such that $x_1 \in B_S(x, 1)$ and $x_1 \neq x$. Now we choose $x_2 \in A$ such that $x_2 \in B_S(x, \frac{1}{2})$ and $x_2 \neq x$, $x_2 \neq x_1$. If we continue in this way, we choose $x_n \in A$ such that $x_n \in B_S(x, \frac{1}{n})$ and $x_n \neq x_1, x_n \neq x_2, \dots, x_n \neq x_{n-1}, \dots$. Consequently we obtain a sequence $\{x_n\}$ consisting distinct element of A such that $\lim_{n \rightarrow \infty} S(x_n, x_n, x) = 0$. The proof is completed. ■

THEOREM 4.2. *Let (X, S) be an S -metric space, T be a continuous C_S - mapping on X and x be a point in X for which $\{T^n x\}_{n=0}^{\infty}$ has a cluster point x_0 . Then $T^n x_0, n = 0, 1, 2, \dots$ are cluster points of $\{T^n x\}_{n=0}^{\infty}$.*

Proof. Let x_0 be a cluster point of $\{T^n x\}_{n=0}^{\infty}$. Then there exists a subsequence $\{T^{n_i} x\}$ which converges to x_0 , that is,

$$\lim_{n_i \rightarrow \infty} S(T^{n_i} x, T^{n_i} x, x_0) = 0.$$

We now show that $T^n x_0, n = 0, 1, 2, \dots$ are cluster points of $\{T^n x\}_{n=0}^{\infty}$.

Since T is a continuous C_S -mapping on X ,

$$\lim_{n_i \rightarrow \infty} S(T^{n_i}x, T^{n_i}x, T^n x_0) \leq \lim_{n_i \rightarrow \infty} \max\{S(T^{n_i}x, T^{n_i}x, x_0)\} = 0.$$

Consequently $T^n x_0$, $n = 0, 1, 2, \dots$ are cluster points of $\{T^n x\}_{n=0}^\infty$. ■

THEOREM 4.3. *Let (X, S) be an S -metric space, T be a continuous C_S -mapping on X and x be a point in X for which $\{T^n x\}_{n=0}^\infty$ has a cluster point x_0 . Then T has a fixed point in $\{T^n x_0\}_{n=0}^\infty$ if and only if one of the following is satisfied:*

- (1) $\{T^n x\}_{n=0}^\infty$ is a convergent sequence.
- (2) There exists a positive integer q such that $T^q z = z$, where z is some point in $\{T^n x_0\}_{n=0}^\infty$.

Proof. If $\{T^n x_0\} = \{x_0\}$, then it can be easily seen that $\{T^n x\}$ is convergent and the condition (1) is satisfied. Let $\{T^n x_0\} \neq \{x_0\}$ and $z \in \{T^n x_0\}$ be a fixed point of T . Since z is a cluster point of $\{T^n x\}$, there exists a subsequence $\{T^{n_i} x\}$ which converges to z . Thus by Theorem 4.2 we obtain

$$\lim_{n_i \rightarrow \infty} S(T^{n_i} x, T^{n_i} x, z) = \lim_{n_i \rightarrow \infty} S(T^{n_i} x, T^{n_i} x, T^n z).$$

Then we have $T^n z = z$ and so the condition (2) is satisfied.

Conversely, if the condition (1) is satisfied, then $\{T^n x_0\} = \{x_0\}$ and x_0 is a fixed point. If the condition (2) is satisfied, then T has a fixed point by Theorem 3.1. ■

COROLLARY 4.1. *Let (X, S) be an S -metric space, T be a continuous L_S -mapping (or T satisfies the condition (S25)) on X and x be a point in X for which $\{T^n x\}_{n=0}^\infty$ has a cluster point x_0 . Then T has a fixed point in $\{T^n x_0\}_{n=0}^\infty$ if and only if one of the following is satisfied:*

- (1) $\{T^n x\}_{n=0}^\infty$ is a convergent sequence.
- (2) There exists a positive integer q such that $T^q z = z$, where z is some point in $\{T^n x_0\}_{n=0}^\infty$.

5. Some generalizations of Nemytskii-Edelstein's and Ćirić's fixed point results

In this section, we give new generalizations of the classical Nemytskii-Edelstein and Ćirić's fixed point theorems for continuous self-mappings of a compact S -metric space. At first we recall the following contractive condition:

$$d(Tx, Ty) < d(x, y),$$

for all $x, y \in X$ with $x \neq y$, where (X, d) is a metric space and T a self-mapping on X .

We note that the completeness of a metric space is not sufficient to guarantee the existence of a fixed point if the contractive condition in Banach fixed point

theorem is replaced by the above contractive condition. Nemytskii [6] and (independently) Edelstein [3] proved a fixed point theorem using the above contractive condition. Hence, the following theorem is known as the classical Nemytskii-Edelstein fixed point theorem.

THEOREM 5.1. [3, 6] *Let T be a mapping from a compact metric space (X, d) into itself satisfying*

$$d(Tx, Ty) < d(x, y),$$

for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point.

In [12], a generalization of the above classical theorem was given as follows:

THEOREM 5.2. [12] *Let (X, S) be a compact S -metric space with $T : X \rightarrow X$ satisfying*

$$S(Tx, Tx, Ty) < S(x, x, y), \quad (5.1)$$

for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point.

As a result of Theorem 3.4, we give now two new generalizations of Nemytskii-Edelstein theorem for continuous self-mappings on a compact S -metric space. Notice that if T satisfies the inequality (5.1) then T satisfies the condition (S25). Indeed we have

$$\begin{aligned} S(Tx, Tx, Ty) &< S(x, x, y) \\ &< \max\{S(x, x, y), S(Tx, Tx, x), S(Ty, Ty, y), S(Ty, Ty, x), S(Tx, Tx, y)\}, \end{aligned}$$

for all $x, y \in X$ with $x \neq y$. Therefore we can deduce the following:

- (1) Corollary 3.3 is a generalization of Theorem 5.2.
- (2) Using Proposition 2.3 it follows that Theorem 3.4 is another generalization of Theorem 5.2.

Now we give an example of a continuous self-mapping which satisfies the conditions (S25) and (S25a) but does not satisfy the inequality (5.1).

EXAMPLE 5.1. Let $X = [0, 1]$ with the usual S -metric given in Example 2.1. Let us define the function $T : X \rightarrow X$ as

$$Tx = \begin{cases} x + \frac{1}{2}, & x \in [0, \frac{1}{2}) \\ 1, & x \in [\frac{1}{2}, 1] \end{cases}$$

for all $x \in X$. Then T is a continuous self-mapping on the compact S -metric space $([0, 1], S)$.

For $x, y \in [0, \frac{1}{2})$ we have

$$S(Tx, Tx, Ty) = 2|x - y| < S(x, x, y) = 2|x - y|.$$

This shows that the inequality (5.1) is not satisfied. It can be easily seen that T satisfies the conditions (S25) and (S25a). Consequently, T has a unique fixed point $x = 1$ in $[0, 1]$.

Recently, Ćirić's fixed point result was also generalized by Sedghi and Dung as seen in the following corollary.

COROLLARY 5.1. [13] *Let (X, S) be a complete S -metric space and T be a self-mapping on X satisfying*

$$S(Tx, Tx, Ty) \leq h \max\{S(x, x, y), S(Tx, Tx, x), S(Ty, Ty, y), \\ S(Ty, Ty, x), S(Tx, Tx, y)\}, \quad (5.2)$$

for all $x, y \in X$ and some $h \in [0, \frac{1}{3})$. Then T has a unique fixed point in X and T is continuous at the fixed point.

Finally, we note that Corollary 3.3 is also a generalization of Corollary 5.1. In [8], the present authors called the inequality (5.2) as (Q25) and then gave another generalization of Corollary 5.1 for continuous self-mappings on a compact S -metric space. Also, this last generalization coincides with Corollary 3.3. If we consider the self mapping defined in Example 5.1 then it can be easily checked that the inequality (5.2) is not satisfied.

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